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STRUCTURE OF A TURBULENT BORE IN A HOMOGENEOUS LIQUID

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A mathematical model for the propagation of nonlinear long waves is constructed with allowance for nonhydrostatic pressure distribution and the development of a surface boundary layer due to wave breaking. The problem of the structure of a bore in a homogeneous liquid is solved. In particular, the transition of a wave bore to a turbulent bore as its amplitude increases is described within a single model.

Introduction. The problems of the structure of stationary hydraulic jumps and bores propagating at constant velocity are classical problems in the hydraulics of open waterways. The key problem in the description of finite-amplitude waves involves modeling of wave breaking and development of a surface turbulent layer. This problem was studied theoretically and experimentally in [1–10].

In the present paper, the approach of [11, 12] for two-layer miscible liquids is used to study the evolution of breaking waves in a homogeneous liquid. The main idea here consists in using the full laws of conservation of mass, momentum, and energy to describe the dynamics of a surface turbulent layer, and it is assumed that the rate of drawing of the liquid from the lower layer depends on the turbulence level in the upper layer. A similar approach was employed in [9, 10], where the conditions of equilibrium between the generation and dissipation of turbulent energy in the upper layer are used instead of the equation of mass inflow into the turbulent layer. Therefore, the model of [9, 10] is an equilibrium model for the system of equations studied below with hydrostatic pressure distribution at the wave front.

The effects due to nonhydrostatic pressure distribution arise primarily for waves of moderate amplitude and lead to the wave structure of a bore. In the majority of models, the effect of nonhydrostatic pressure distribution on the wave shape is described using various versions of the Boussinesq and Korteweg–de Vries equations that result from the second shallow-water approximation [13].

In the present model, the terms describing the effects of nonhydrostatic pressure distribution are included in the equations in an unusual manner. The system of equations is hyperbolic, and the drawing of the liquid from the lower into the turbulent layer prevents the system from leaving the hyperbolic region. The model describes the formation of solitonlike waves propagating in a liquid at rest and the structure of a wave bore and its transition to a turbulent bore as the bore amplitude increases.

Mathematical Model. We consider planar flow of a thin layer of an ideal incompressible liquid over a horizontal bottom under gravity. The turbulent mixing caused by wave breaking is described by the equations of two-layer, shallow water [11]. Aeration of the flow is ignored, and, hence, the liquid density ρ is constant. The lower layer, in which the flow is considered potential, is characterized by two parameters: the mean depth h and the nonzero mean horizontal velocity u in the layer. The turbulent upper layer is specified by its depth η, velocity v, and the root-mean-square velocity q of the turbulent flow. Under the assumption of hydrostatic pressure distribution in the layers, the equations of motion are written as

$$(h+\eta)_t + (hu+\eta v)_x = 0, \quad u_t + (u^2/2 + g(h+\eta))_x = 0,$$

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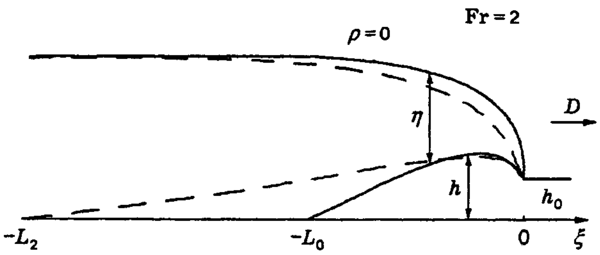

Fig. 1

$$(hu+\eta v)_t+\left(hu^2+\eta v^2+\frac{1}{2}g(h+\eta)^2\right)_x=0, \tag{1.1}$$

$$(hu^2+\eta(v^2+q^2)+g(h+\eta)^2)_t+(hu^3+\eta(v^2+q^2)v+2g(h+\eta)(hu+\eta v))_x=-\beta q^3, \quad \eta_t+(\eta v)_x=\sigma q.$$

System (1.1) contains the equations of two-layer shallow water written in the form of integral laws of conservation. The energy equation is used to determine the law of drawing of the liquid from the lower layer in the turbulent layer, and the rate of drawing is assumed to be proportional to the velocity q. The proportionality coefficient $\sigma = 0.15$ [14] characterizes the ratio of the vertical and horizontal scales of motion and does not affect the wave structure since it can be excluded from the equations by extending the independent variables. The constant β specifies the rate of dissipation of turbulent kinetic energy.

The structure of Eqs. (1.1) becomes clearer if they are written in nondivergent form

$$\begin{gathered} h_t+uh_x+hu_x=-\sigma q, \qquad \eta_t+v\eta_x+\eta v_x=\sigma q, \qquad u_t+uu_x+gh_x+g\eta_x=0, \\ v_t+vv_x+gh_x+g\eta_x=\frac{\sigma q}{\eta}(u-v), \qquad q_t+vq_x=\frac{\sigma}{2\eta}((u-v)^2-(1+\delta)q^2), \end{gathered} \tag{1.2}$$

where $\delta=\beta/\sigma\equiv \text{const}$.

Thus, system (1.2) or (1.1) contains the only essential dimensionless parameter $\delta>0$, which should be chosen empirically. As shown below, system (1.2) is adequate for describing the structure of developed turbulent bores in a homogeneous liquid.

2. Structure of a Turbulent Bore. We consider finite-amplitude disturbances that propagate with constant velocity in an initially quiescent liquid. The initial depth of the layer is $h=h_0$, the velocity is $u_0=0$, and a turbulent interlayer is absent ($\eta_0=0$). It is required to describe traveling waves, i.e., waves of constant shape propagating at constant velocity D. The only dimensionless parameter of the problem is the Froude number $\text{Fr}=D/\sqrt{gh_0}$ (Fig. 1).

It is known [13] that in a shallow-water approximation ignoring mixing ($\sigma=0$), a traveling wave (bore) at all values of $\text{Fr}>1$ represents a discontinuous solution that transforms the layer at rest into a homogeneous layer of constant depth moving at constant velocity. On the discontinuity line, the laws of conservation of mass and momentum are fulfilled, and the total energy of the flow decreases.

Solution of the formulated problem using model (1.1) leads to a paradox due to the simultaneous use of the laws of conservation of mass, momentum, and energy in (1.1). Indeed, at first glance, a continuous solution different from the trivial solution for $\text{Fr}>1$ cannot exist since the wave front moves at supercritical velocity. On the other hand, in a discontinuous solution, by virtue of the relations at the jump, the interlayer thickness directly behind the discontinuity must equal zero, and, therefore, the laws of conservation of mass, momentum, and energy cannot be fulfilled simultaneously for jumps of finite amplitude.

As shown below, this paradox can be resolved by closer examination of the structure of flows with a turbulent layer. When the flow parameters in the upper layer near the front vary continuously, the liquid is accelerated, acquiring almost the wave velocity ($v\sim D$). Thus, the flow characteristics overtake the wave front, and the structure of the turbulent bore can be described in the class of continuous solutions of system (1.1).

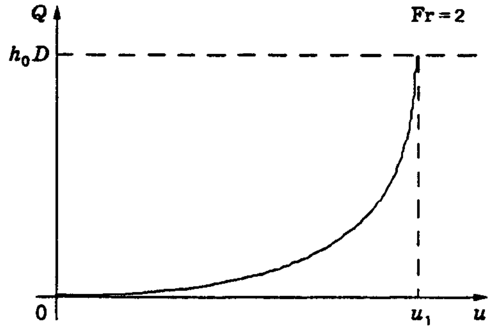

Fig. 2

Let the solution of system (1.1) depend only on the variable $\xi = x - Dt$, where $D > 0$. The homogeneous laws of conservation in (1.1) yield the relations

$$h(u-D)+\eta(v-D)=-h_0D, \qquad \frac{1}{2}(u-D)^2+g(h+\eta)=\frac{1}{2}D^2+gh_0,$$
$$hu(u-D)+\eta v(v-D)+\frac{1}{2}g(h+\eta)^2=\frac{1}{2}gh_0^2. \tag{2.1}$$

In the dimensionless variables ($h_0 = 1$ and $g = 1$), the quantities h, η, and v are defined in the region of admissible values ($\eta > 0$ and $h > 0$) as functions of the variable u ($u > 0$) and the parameter $\mathrm{Fr} = D/\sqrt{gh_0} = D > 1$. In the limit $u \to 0$, we have $h \to h_0$ and $\eta \to 0$ by virtue of the continuity of the solution, but in the limit $\eta \to 0$, system (2.1) strongly degenerates. The limiting values v_0 are obtained by differentiating system (2.1) with respect to one of the variables, for example, h. We have two cases:

(a) $v_0 = 0$; (b) $v_0 = D$.

Case (a) corresponds to the following exact solution of (2.1): $h + \eta \equiv h_0$, $u = v \equiv 0$. In case (b), the behavior of the solution of (2.1) in the plane (u, Q), where $Q = \eta(D - v)$, is illustrated by Fig. 2 for $\mathrm{Fr} = 2$. The limiting velocity u_1 in the wave is attained for $Q = h_0D$, for which the turbulent layer reaches the bottom. Drawing ceases, the depth and velocity in the wave remain constant, and the turbulence level decreases, i.e., the turbulence degenerates throughout the depth of the liquid.

Let us consider in more detail the behavior of the root-mean-square velocity q of the turbulent flow in the transition region of the wave ($0 < Q < h_0D$). It follows from Eqs. (1.1) that the variables entering these system can be expressed as functions of one variable, for example, $Q = \eta(D - v)$ $(0 < Q < h_0D)$. Then, the inhomogeneous conservation laws (1.1) lead to the following system of ordinary differential equations for traveling waves:

$$Q' = -\sigma q, \qquad Qq' = \frac{\sigma}{2}((1+\delta)q^2 - f(Q)). \tag{2.2}$$

Here $f(Q) = (u - v)^2(Q)$, and the prime denotes differentiation with respect to ξ. The solution of this autonomous system can be found in quadratures since in the phase plane (Q, q^2), system (2.2) reduces to the linear equation

$$Q\frac{dq^2}{dQ} = f(Q) - (1+\delta)q^2. \tag{2.3}$$

The limited solution of (2.3) has the form

$$q^2(Q) = Q^{-(1+\delta)}\int_0^Q s^\delta q(s)\,ds, \tag{2.4}$$

and, hence, the limiting value $q^2(0)$ at the wave hollow is

$$q^2(0) = f(0)/(1+\delta).$$

In case (a), $q^2 \equiv 0$ and the initial state is undisturbed. In case (b), $q^2(0) = v^2(0)/(1+\delta) = D^2/(1+\delta)$.

Thus, mixing at constant rate begins at the wave hollow. The wave profile given by the functions $h = h(\xi)$ and $\eta = \eta(\xi)$ can be found by virtue of (2.1) and (2.4) from the equation

$$\frac{dQ}{d\xi} = \sigma q(Q).$$

According to (2.1), in a neighborhood of $u = 0$ the representation

$$\eta = \frac{(D^2 - gh_0)u}{gD} + o(u), \quad v = D - \frac{1}{2}u + o(u), \quad Q = \frac{(D^2 - gh_0)}{2gD}u^2 + o(u^2), \quad q = D + O(u)$$

is valid, and, hence, near the point $\xi = 0$, which corresponds to the position of the front, the functions $h(\xi)$ and $\eta(\xi)$ have an integrable singularity [$h(\xi) \sim |\xi|^{1/2}$ and $\eta(\xi) \sim |\xi|^{1/2}$ for $\xi < 0$].

The structure of a turbulent bore in a homogeneous liquid for Fr = 2 is depicted in Fig.1, where the solid and dashed curves show the boundaries of the turbulent layer for $\delta = 0$ and $\delta = 2$, respectively.

For Fr > 1, a continuous monotonic transition to another level η_1 occurs in a turbulent bore within a zone of finite length L ($L = L_0$ for $\delta = 0$, and $L = L_2$ for $\delta = 2$). As Fr increases, the length of the transition zone decreases. Clearly, the states in front of and behind the wave are related by ordinary shallow-water relations, which follows from the laws of conservation of mass and momentum [cf. (2.1)]:

$$\eta_1(v_1 - D) = -h_0 D, \qquad \eta_1 v_1(v_1 - D) + \frac{1}{2}g\eta_1^2 = \frac{1}{2}gh_0^2,$$

and the energy equation is an additional condition that allows one to separate out physically admissible solutions with $q_1^2 \geqslant 0$.

Summarizing the aforesaid, we can state that inclusion of mixing in the shallow-water equations allows us to solve the problem of the structure of a hydraulic jump in a homogeneous liquid. A drawback of the model is the hypothesis on hydrostatic pressure distribution throughout the zone of the jump, which prevents manifestation of the wave nature of transition for Froude numbers close to unity. Below, the model is developed taking into account nonhydrostatic effects. In Sec. 3, a simpler hyperbolic shallow-water model than the one in [15] is derived with allowance for dispersion effects.

3. Shallow-Water Equations with Dispersion. *Equations of Motion.* The equations of one-layer shallow water are derived for mean values of the depth $h(t,x)$ and velocity $u(t,x)$ of the homogeneous layer. In Secs. 1 and 2, the influence of surface waves and mixing on the flow structure was taken into account by introducing a turbulent interlayer. In this case, the pressure distribution was considered hydrostatic. This hypothesis is valid for waves and vortices whose scale is far less than the layer thickness. Propagation of disturbances of moderate amplitude in a liquid at rest gives rise to a packet of waves whose wavelength is comparable with the thickness of the homogeneous layer. That is, a wave (undulated) bore forms, and the nonhydrostatic pressure distribution at the wave front plays an important part in the formation of the bore. The structure of the wave bore is described using various versions of the Boussinesq and Korteweg–de Vries equations that correspond to the second shallow-water approximation [13]. The dispersion effects are modeled by terms that contain higher-order derivatives, so that the second approximation is no longer defined by the hyperbolic system of equations.

The hyperbolic model of a dispersive medium arises when, along with mean flow characteristics such as depth $h(t,x)$ and velocity $u(t,x)$, internal variables, which characterize the state of flow at a point of interest, are used to describe nonequilibrium processes under gravity (Fig. 3). In this case, equal values of the mean and instantaneous parameters indicate an equilibrium state of the flow.

Let $\zeta(t,x)$ be the instantaneous depth and $w(t,x)$ the vertical velocity of the liquid at the surface. The difference between the mean depth $h(t,x)$ (solid curve) and the instantaneous depth $\zeta(t,x)$ (dashed curve) of the homogeneous-liquid layer arises when a wave with a wavelength comparable with the spatial scale of

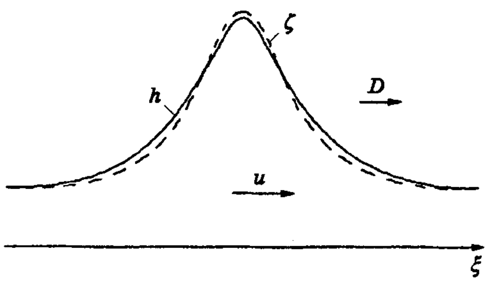

Fig. 3

averaging used to obtain the mean depth $h(t,x)$ travels in the liquid in the vicinity of the point x. In the averaged equations of motion, the effect due to the nonhydrostatic pressure distribution caused by short-wave generation at the fronts of nonlinear disturbances of the flow can be taken into account by means of the function p^*, which gives the deficit of pressure on the bottom, as follows

$$h_t + (hu)_x = 0, \qquad u_t + uu_x + gh_x + p^*_x = 0. \tag{3.1}$$

The nonhydrostatic pressure distribution in the liquid layer leads to the necessity of invoking the momentum equation for the vertical velocity in order to determine the function $\zeta = \zeta(t,x)$. In what follows, we assume that $|\zeta/h - 1| \ll 1$. The equations for ζ and the vertical velocity w at the surface take the form

$$\zeta_t + u\zeta_x = w, \qquad h(w_t + uw_x) = 2p^*. \tag{3.2}$$

Here the first equation gives the kinematic condition at the liquid surface, and the second is obtained by integration of the momentum equation for the vertical velocity over the entire layer with allowance for the nonpenetration condition at the bottom and the linear distribution of the vertical velocity across the layer depth.

To close system (3.1), (3.2), it is required to express p^* in terms of the required variables. By virtue of the smallness of the difference $h - \zeta$ compared to h and from the conditions of equilibrium $p^*(h,h) \equiv 0$, the dependence

$$p^* = \alpha g(h - \zeta) \tag{3.3}$$

seems to be rather general. The parameter α serves to fix the chosen scale of averaging in the model. As $\alpha \to 0$, system (3.1) becomes independent of (3.2) and reduces to the standard shallow-water equations. This limiting case corresponds to a rather large averaging scale, and the other limit $\alpha \to \infty$ corresponds to a decrease in the averaging scale. In this case, $\zeta \equiv h$, and the variable p^* becomes nonevolutionary. The system obtained is equivalent to a Boussinesq equation that corresponds to the second shallow-water approximation. Thus, for finite values of α, system (3.1)–(3.3) gives a shallow-water approximation that is intermediate between the first and the second approximations. It describes the dispersive properties as the second shallow-water approximation and preserves hyperbolicity as the first approximation. The choice of the parameter α in the equations corresponds to the choice of the averaging scale in the description of the waves generated at the front of a nonlinear bore. We note that system (3.1)–(3.3) is similar in structure to the Iordanskii single-velocity model for a monodisperse bubble liquid [16].

The characteristics of system (3.1)–(3.3) have the form

$$\frac{dx}{dt} = \lambda^{\pm} = u \pm \sqrt{(1+\alpha)gh}.$$

In addition, there is the multiple contact characteristic

$$\frac{dx}{dt} = \lambda_0 = u.$$

The system in equilibrium ($h \equiv \zeta$) coincides with the ordinary shallow-water equations, and, hence, the characteristics of the system in equilibrium are

$$\frac{dx}{dt} = \lambda_e^{\pm} = u \pm \sqrt{gh}.$$

Thus, in analyzing the wave structure in model (3.1)–(3.3), we are dealing with the classical case where the characteristics of the original and equilibrium models alternate: $\lambda^- < \lambda_e^- < \lambda_0 < \lambda_e^+ < \lambda^+$, and, hence, smooth traveling waves (solitons) for the full system of equations exist in the velocity ranges $\lambda_e^+ < D < \lambda^+$ and $\lambda^- < D < \lambda_e^-$ (see [13]).

Traveling Waves Given by System (3.1)–(3.3). We assume that the solutions depend only on the variable $\xi = x - Dt$ ($D > 0$), and as $\xi \to \infty$, the solution tends to the equilibrium solution: $h_0 = \zeta_0$ and $u_0 = w_0 = 0$. As in the previous case, the only dimensionless parameter that characterizes the wave is the Froude number $\mathrm{Fr} = D/\sqrt{gh_0}$. For a traveling wave, system (3.1)–(3.3) takes the form

$$(h(u-D))' = 0, \qquad \Big(\frac{1}{2}u^2 - Du + (1+\alpha)gh - \alpha g\zeta\Big)' = 0,$$
$$(u-D)\zeta' = w, \qquad h(u-D)w' = 2\alpha g(h-\zeta). \tag{3.4}$$

Here the prime denotes differentiation with respect to the variable ξ. System (3.4) leads to the relations

$$h(u-D) = -h_0 D, \qquad \frac{1}{2}u^2 - Du + (1+\alpha)gh - \alpha g\zeta = gh_0, \tag{3.5}$$

from which the functions $u = u(h)$ and $\zeta = \zeta(h)$ can be obtained:

$$u(h) = \frac{D(h-h_0)}{h}, \qquad \zeta(h) = h - \frac{G(h)}{\alpha g}. \tag{3.6}$$

Here $G(h) = p^*(h) = g(h-h_0) + D^2(h^2 - h_0^2)/(2h^2)$.

We introduce the designation $a(h) = d\zeta(h)/dh = 1 - (D^2h_0^2/h^3 - g)/(\alpha g)$. With allowance for (3.5) and (3.6), system (3.4) takes the form

$$D^2h_0^2\Big(\frac{a(h)}{h}h'\Big)' = 2G(h). \tag{3.7}$$

Multiplying Eq. (3.7) by $a(h)h'/h$, we obtain the integral

$$\frac{1}{2}D^2h_0^2\Big(\Big(\frac{a(h)h'}{h}\Big)^2\Big)' = (F(h))' = \frac{2G(h)a(h)h'}{h}. \tag{3.8}$$

The behavior of the function

$$F(h) = \int_{h_0}^{h} \frac{2G(s)a(s)\,ds}{s}$$

in the vicinity of h_0 determines the structure of the steady solution of (3.4). The function $F(h)$ can be expressed via elementary functions, but to determine the domain of parameters in which solitons exist, it suffices to clarify the qualitative behavior of this function in the vicinity of h_0. The traveling wave profile can be found from (3.8) in quadratures:

$$\xi = \xi_0 \pm \int_{h_0}^{h} \frac{Dh_0 a(s)\,ds}{s\sqrt{2F(s)}}. \tag{3.9}$$

The function $a(h) = d\zeta(h)/dh$ vanishes at a single point h_*, which corresponds to the minimum of the function $\zeta(h)$. We note that $h_* > h_0$ if $D > \sqrt{(1+\alpha)gh_0}$. From relation (3.9) it follows that the necessary condition

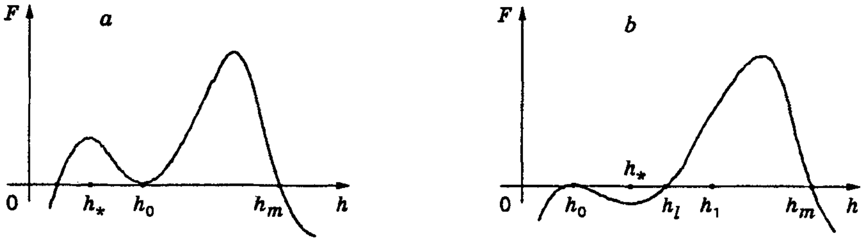

Fig. 4

for the existence of solution (3.8) is the positiveness of the function $F(h)$ in the vicinity of $h = h_0$. Since

$$\frac{dF}{dh} = \frac{2\alpha g(h - \zeta(h))a(h)}{h},$$

the inequality $0 < a(h_0) < 1$, which ensures the positiveness of the function $F(h)$ near the point h_0, is fulfilled if

$$\sqrt{gh_0} < D < \sqrt{(1+\alpha)gh_0}. \tag{3.10}$$

Figure 4a gives a plot of the function $F(h)$ for the velocity D which satisfies inequalities (3.10). For $h > h_0$, the solution of system (3.4) is a soliton whose velocity is between the equilibrium and frozen propagation velocities of the characteristics, and the maximum depth is h_m. The distribution of the mean depth $h(\xi)$ and instantaneous depth $\zeta(\xi)$ in the soliton is shown in Fig. 3. Recall that the difference $h(\xi) - \eta(\xi)$ varies in proportion to the deficit of the wave pressure p^*. For $h < h_0$, the behavior of the function $F(h)$ is similar to the case $h > h_0$. This branch, however, does not give a soliton since, in passage through the point $h = h_*$, the function $a(h)$ changes sign, and solution (3.9) cannot be continued to the region $h < h_*$.

A smooth soliton of limiting amplitude occurs for $D = \sqrt{(1+\alpha)gh_0}$ ($\mathrm{Fr} = \sqrt{1+\alpha}$). For $D > \sqrt{(1+\alpha)gh_0}$, only a configuration of the jump–wave type is possible [16]. This configuration consists of a hydraulic jump (bore) which transforms a layer of depth h_0 into a layer of depth h_1 ($h_l \leqslant h_1 \leqslant h_m$) for $F(h_1) \geqslant 0$, and the adjacent periodic solution with maximum and minimum depths h_l and h_m, respectively (Fig. 4b).

Since Eqs. (3.1)–(3.3) are written in nondivergent form, the choice of hydraulic jump relations that determine the value of h_1 behind the front is a separate problem and is not considered here. We note only that simultaneous consideration of dispersion and mixing (see Sec. 4) permits constructing a continuous wave profile for $D > \sqrt{(1+\alpha)gh_0}$ as well.

4. Shallow-Water Equations with Dispersion and Mixing. To combine the two approaches to modeling the effects of mixing and dispersion developed in Secs. 2 and 3, it suffices to assume that in a turbulent layer there is hydrostatic pressure distribution due to developed small-scale motion, and nonhydrostatic pressure distribution is manifested only in the lower homogeneous layer. Thus, system (3.1)–(3.3) is in fact supplemented by the equations of motion of a turbulent interlayer taking into account drawing.

Equations of motion. The full system of equations takes the form

$$h_t + (hu)_x = -\sigma q, \quad \eta_t + (\eta v)_x = \sigma q, \quad u_t + uu_x + (1+\alpha)gh_x + g\eta_x - \alpha g\zeta_x = 0,$$
$$v_t + vv_x + g(h_x + \eta_x) = \frac{\sigma q}{\eta}(u - v), \quad q_t + vq_x = \frac{\sigma}{2\eta}((u-v)^2 - (1+\delta)q^2), \tag{4.1}$$
$$\zeta_t + u\zeta_x = w, \qquad w_t + uw_x = \frac{2\alpha g(h - \zeta)}{h}.$$

System (4.1) describes the flow of a homogeneous layer with a turbulent interlayer. Equations (1.2) are obtained from (4.1) in the limit $\alpha \to 0$, and Eqs. (3.1)–(3.3) arise when $\eta \to 0$ and $\sigma = 0$.

Despite the appreciable extension of system (4.1) compared to (1.2), the introduction of new required variables does not affect the hyperbolicity of the system. Moreover, the characteristics of (4.1) coincide with the characteristics of the system of equations for two-layered shallow water with densities ρ and $(1+\alpha)\rho$ in the two layers (see [14]). Introduction of the variables ζ and w into system (4.1) yields only the multiple characteristic $dx/dt = u$. Interestingly, for a homogeneous liquid, the characteristics of system (4.1) do not degenerate as happens with (1.2) since the effective density $(1+\alpha)\rho$ of the lower layer no longer coincides with the density ρ of the interlayer. Therefore, the nonhydrostatic pressure distribution at the wave front exerts a certain stabilizing effect early in the development of the interlayer. Next, we shall discuss details of the solution of the problem of the structure of a traveling wave in a homogeneous liquid within the framework of the model (4.1).

Structure of a Turbulent Bore in a Homogeneous Liquid. We consider the traveling waves given by system (4.1). Let the solution depend on the variable $\xi = x - Dt$ $(D > 0)$. Designating derivatives with respect to ξ by primes, we obtain the system

$$\begin{gathered}(h(u-D))' = -\sigma q, \qquad (\eta(v-D))' = \sigma q, \qquad (u-D)u' + (1+\alpha)gh' + g\eta' - \alpha g\zeta' = 0,\\ (v-D)v' + g(h'+\eta') = \frac{\sigma q}{\eta}(u-v), \qquad (v-D)q' = \frac{\sigma}{2\eta}((u-v)^2 - (1+\delta)q^2),\\ (u-D)\zeta' = w, \qquad h(u-D)w' = 2\alpha g(h-\zeta).\end{gathered} \tag{4.2}$$

It is required to find a continuous solution of (4.2) which, as $\xi \to \infty$, has the following limits: $h \to h_0$, $\zeta \to h_0$, $\eta \to 0$, $u \to 0$, $w \to 0$, $v \to 0$, and $q \to 0$. System (4.2) strongly degenerates as $\eta \to 0$. Hence, to obtain the asymptotic behavior of the solution for high values of ξ, we consider the semilinear system derived from (4.2) by linearization of the left-hand side on the degenerated solution $h = h_0$, $\zeta = h_0$, $\eta = 0$, $u = 0$, $w = 0$, $v = 0$, and $q = 0$. The right-hand side of Eqs. (4.2) remains unchanged:

$$\begin{gathered}h_0\tilde{u}' - D\tilde{h}' = -\sigma\tilde{q}, \qquad -D\tilde{\eta} = \sigma\tilde{q}, \qquad -D\tilde{u}' + (1+\alpha)g\tilde{h}' + g\tilde{\eta}' - \alpha g\tilde{\zeta}' = 0,\\ -D\tilde{v}' + g\tilde{h}' + g\tilde{\eta}' = \frac{\sigma\tilde{q}(\tilde{u}-\tilde{v})}{\tilde{\eta}}, \qquad -D\tilde{q}' = \frac{\sigma((\tilde{u}-\tilde{v})^2 - (1+\delta)\tilde{q}^2)}{2\tilde{\eta}},\\ -D\tilde{\zeta}' = \tilde{w}, \qquad -Dh_0\tilde{w}' = 2\alpha g(\tilde{h}-\tilde{\zeta}).\end{gathered} \tag{4.3}$$

Small perturbations of the main flow are denoted by the tilde. It should be noted that system (4.3) is homogeneous with respect to the perturbations, and, hence, its solutions are functions of the form

$$\tilde{h} = \hat{h}e^{-\lambda\xi}, \quad \tilde{\zeta} = \hat{\zeta}e^{-\lambda\xi}, \quad \tilde{\eta} = \hat{\eta}e^{-\lambda\xi}, \quad \tilde{u} = \hat{u}e^{-\lambda\xi}, \quad \tilde{v} = \hat{v}e^{-\lambda\xi}, \quad \tilde{q} = \hat{q}e^{-\lambda\xi}, \quad \tilde{w} = \hat{w}e^{-\lambda\xi}$$

for an appropriate positive value of λ. The parameter λ is found from the algebraic system of equations

$$\begin{gathered}\lambda h_0\hat{u} - \lambda D\hat{h} = \sigma\hat{q}, \qquad \lambda D\hat{\eta} = \sigma\hat{q}, \qquad \lambda(D\hat{u} - (1+\alpha)g\hat{h} - g\hat{\eta} + \alpha g\hat{\zeta}) = 0,\\ \lambda D\hat{v} - \lambda g\hat{h} - \lambda g\hat{\eta} = \frac{\sigma\hat{q}(\hat{u}-\hat{v})}{\hat{\eta}}, \quad \lambda D\hat{q} = \frac{\sigma((\hat{u}-\hat{v})^2 - (1+\delta)\hat{q}^2)}{2\hat{\eta}},\\ \lambda D\hat{\zeta} = \hat{w}, \qquad \lambda Dh_0\hat{w} = 2\alpha g(\hat{h}-\hat{\zeta}).\end{gathered} \tag{4.4}$$

Let $\sigma \neq 0$ and $\lambda \neq 0$. Relations (4.4) can be rearranged as

$$\begin{gathered}h_0\hat{u} - D\hat{h} = D\hat{\eta}, \quad D\hat{u} - (1+\alpha)g\hat{h} - g\hat{\eta} + \alpha g\hat{\zeta} = 0, \quad D\hat{v} - g\hat{h} - g\hat{\eta} = D(\hat{u}-\hat{v}),\\ 2\lambda D\hat{\eta}\hat{q} = \sigma((\hat{u}-\hat{v})^2 - (1+\delta)\hat{q}^2), \quad \lambda D\hat{\eta} = \sigma\hat{q}, \quad \lambda^2 D^2 h_0\hat{\zeta} = 2\alpha g(\hat{h}-\hat{\zeta}).\end{gathered} \tag{4.5}$$

System (4.5) is nonlinear, and its trivial solution is $\hat{h} = \hat{\eta} = \hat{\zeta} = \hat{u} = \hat{v} = \hat{q} = 0$. To obtain the spectral relationship for λ for which there is a nontrivial physically admissible solution ($\hat{\eta} > 0$, $\hat{h} > 0$, $\hat{\zeta} > 0$, $\hat{q} > 0$, and $\lambda > 0$), it suffices to express all required variables through one of them. From (4.5), we have

$$\hat{\zeta} = \frac{2\alpha g\hat{h}}{2\alpha g + \lambda^2 D^2 h_0} < \hat{h}, \qquad \hat{h} = \frac{(gh_0 - D^2)\hat{\eta}}{D^2 - (1+\alpha)gh_0 + \gamma}, \qquad \hat{v} = \frac{(gh_0 + D^2)(\hat{h}+\hat{\eta})}{2Dh_0},$$

$$\hat{u} = \hat{v} + \frac{(D^2 - gh_0)(\hat{h} + \hat{\eta})}{2Dh_0}, \qquad \hat{q} = \frac{\lambda D\hat{\eta}}{\sigma}, \qquad (3+\delta)\lambda^2 D^2 \hat{\eta}^2 = \sigma^2(\hat{u} - \hat{v})^2, \tag{4.6}$$

where $\gamma = 2\alpha^2 g^2 h_0/(2\alpha g + \lambda^2 D^2 h_0)$. Substituting the expressions of $\hat{u}$ and $\hat{v}$ in terms of $\hat{\eta}$ into the last relation of (4.6), we obtain the following equation for λ:

$$\frac{(D^2 - gh_0)(\gamma - \alpha g h_0)}{D^2 h_0 (D^2 - (1+\alpha)gh_0 + \gamma)} = \frac{\lambda}{p},$$

where $p = \pm\sigma/(2\sqrt{3+\delta})$. By virtue of (4.6), the quantity γ also depends on λ. Since only the value $\lambda > 0$ is admissible, the final form of the equation for λ is

$$\lambda^2 D^2 h_0 (D^2 - (1+\alpha)gh_0) - \lambda p \alpha g h_0 (gh_0 - D^2) + 2\alpha g (D^2 - gh_0) = 0.$$

In the limit $\sigma \to 0$ $(p \to 0)$, we have

$$\lambda = \sqrt{\frac{2\alpha g (D^2 - gh_0)}{D^2 h_0 ((1+\alpha)gh_0 - D^2)}},$$

and a solution of the solitary-wave type (soliton) exists in the range

$$gh_0 < D^2 < (1+\alpha)gh_0, \tag{4.7}$$

as shown in Sec. 3. For $\sigma > 0$, in the range of velocities (4.7), the physically admissible solution with $\hat{\eta} > 0$, $\hat{q} > 0$, $\hat{h} > 0$, and $\hat{\zeta} > 0$ is obtained for $p = \sigma/(2\sqrt{3+\delta})$ and

$$\lambda = \frac{p\alpha g h_0 (gh_0 - D^2)}{2D^2 h_0 (D^2 - (1+\alpha)gh_0)}$$
$$+ \frac{\sqrt{p^2 \alpha^2 g^2 h_0^2 (gh_0 - D^2)^2 - 8\alpha g (D^2 - gh_0) D^2 h_0 (D^2 - (1+\alpha)gh_0)}}{2D^2 h_0 (D^2 - (1+\alpha)gh_0)}. \tag{4.8}$$

Using the asymptotic relation obtained in the limit $\xi \to \infty$, one can find the wave profile by numerical integration of system (4.2). As noted above, for $\sigma \to 0$ and $\eta \to 0$, the wave is a soliton. For $\sigma = 015$, $\alpha = 1$, and Fr $= 1.2$, the wave profile $[h = h(\xi)$ and $h + \eta = (h+\eta)(\xi)]$ is shown in Fig. 5 at $\delta = 0$ (a) and 2 (b).

As in the case of a turbulent bore ($\alpha = 0$), for $\sigma \neq 0$ the interlayer reaches the bottom, and then homogeneous flow is attached to the wave. The wave profile, however, is no longer monotonic. As numerical calculations show, a smooth wave bore exists for Froude numbers $1 < \mathrm{Fr} < \mathrm{Fr}_* < \sqrt{1+\alpha}$ ($\mathrm{Fr}_* \sim 1.3$ for $\alpha = 1$ and $\delta = 0$). Attaining the critical Froude number Fr_*, the determinant of system (4.2) vanishes at the wave front, i.e., the flow with the turbulent interlayer becomes critical at a certain point. The solution cannot be continued in a smooth fashion through the critical point. A weak hydraulic jump originates at the wave front. Analysis of the characteristics shows that this jump corresponds to the second mode, and for $\mathrm{Fr}_* < \mathrm{Fr} < \sqrt{1+\alpha}$, it moves from the crest toward the wave hollow. This reasoning is of a qualitative nature since system (4.2) is written in differential form, and to determine the location and amplitude of the jump uniquely, one should invoke additional relations at the discontinuity. The study of discontinuous solutions is beyond the scope of this work. It is worth noting, however, that the possibility of a hydraulic jump (roller) occurring at the wave crest and moving toward the wave hollow as the amplitude increases is in agreement with the breaking pattern of initially smooth waves observed in laboratory- and full-scale experiments at the moment when their limited amplitude is attained.

We now dwell on the behavior of the dissipation-free solution ($\delta = 0$) for $\mathrm{Fr} > \sqrt{1+\alpha}$. For these Froude numbers, the asymptotic behavior of the wave front changes. Formula (4.8) in this case does not yield real values of λ but there is a solution similar in structure to the turbulent bore considered in Sec. 2 for hydrostatic pressure distribution. Indeed, substituting the solution of system (4.2) as a functional dependence of the required functions, for example, the velocity $u = u(\xi)$, we obtain, as in Sec. 2 that, as $u \to 0$, the

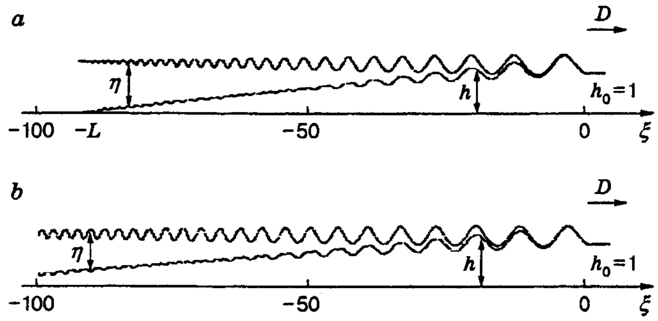

Fig. 5

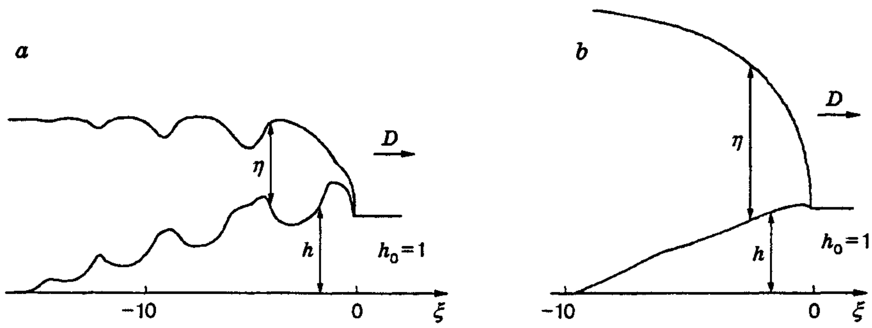

Fig. 6

solution of (4.2) has the limits $h \to h_0$, $\zeta \to \zeta_0$, $\eta \to 0$, $w \to 0$ but $v \to D$, and $q \to D$. At the same time,

$$\left.\frac{dh}{du}\right|_{u=0} = \frac{h_0}{D}, \qquad \left.\frac{d\eta}{du}\right|_{u=0} = \frac{D^2 - (1+\alpha)gh_0}{gD}. \tag{4.9}$$

From the second of relations (4.9) it follows that for $\mathrm{Fr} > \sqrt{1+\alpha}$, the function $\eta(\xi)$ should decrease in the vicinity of the wave front. As in Sec. 2, an integrable singularity arises at the bore front. Without loss of generality, we can assume that the front is localized at the point $\xi = 0$. In this case, $h \sim |\xi|^{1/2}$ and $\eta \sim |\xi|^{1/2}$, and the dependences $h = h(\xi)$ and $\eta = \eta(\xi)$ can be found by integrating Eqs. (4.2) for $\xi < 0$. Numerical calculations for $\alpha = 1$, $\sigma = 0.15$, and $\delta = 0$ show that, in the interval $\sqrt{2} < \mathrm{Fr} < 2$, there is no solution of (4.2) since the determinant of the system vanishes at a certain point $\xi_0 < 0$, where $h(\xi_0) > 0$. For $\mathrm{Fr} > 2$, a smooth solution can be constructed up to the moment the turbulent layer reaches the bottom. The wave profile for $\mathrm{Fr} = 2$ and 3 is shown in Fig. 6a and b, respectively.

For $\mathrm{Fr} < 2.5$, the wave profile is not monotonic but the effect of the nonhydrostatic pressure distribution at the wave front becomes less pronounced as the Froude number increases, and at high values of Fr, the solution approaches the turbulent-bore structure studied in Sec. 2.

Conclusions. The model of a turbulent bore constructed here describes both the nonstationary development of nonlinear waves on the surface of a homogeneous liquid up to the moment of wave breaking and the structure of traveling waves or bores of arbitrary amplitude. For small amplitudes ($\mathrm{Fr} < 1.4$ and $\alpha = 1$), the nonhydrostatic effects play a leading part and a wave bore forms. For $\mathrm{Fr} > 1.4$, a surface turbulent layer develops actively, and for Fr >, 2.5 the bore becomes monotonic.

For $1 < \mathrm{Fr} < \mathrm{Fr}_* \sim 1.3$ and $\mathrm{Fr} > 2$ ($\delta = 0$), there is a continuous solution of system (4.2). In the range $1.3 < \mathrm{Fr} < 2$, a hydraulic jump, presumably of the second mode, develops at the front of the traveling wave. In this paper, the bore structure in this range of Fr was not considered since, to determine the location and amplitude of the hydraulic jump, it is necessary to specify certain relations at discontinuities. For system (4.1), this is impossible since this system is written in nondivergent form. We did not also touch on the effect of dissipation on the bore structure. It was only shown that for $\delta > 0$, the structures of the fronts of the turbulent and wave bores remain qualitatively unchanged compared to the case $\delta = 0$.

Finally, it should be noted that model (1.1) with hydrostatic pressure distribution at the front ($\alpha = 0$) is an extension of the model of [10], where the energy equation was used to determine the interlayer thickness η under the assumption that the production and dissipation of energy in the upper layer are at equilibrium $q^2 \sim (u - v)^2$. In model (1.1), the energy equation is used to determine q^2 and the evolution of the turbulent layer is described by the equation for η [the last equation in (1.1)], in which the rate of drawing of the liquid from the lower layer is proportional to q.

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